, , ,

Extreme charged black holes in braneworld with cosmological constant

Ryotaku Suzuki* and Tetsuya Shiromizu†
Department of Physics, Kyoto University, Kyoto 606-8502, Japan

Norihiro Tanahashi‡
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

Application of the adS/CFT correspondence to the RS models may predict that there is no static solution for black holes with a radius larger than the bulk curvature scale. When the black hole has an extremal horizon, however, the correspondence suggests that the black hole can stay static. We focus on the effects of cosmological constant on the brane on such extremal brane-localized black holes. We observe that the positive cosmological constant restrict the black hole size on the brane as in ordinary four-dimensional general relativity. The maximum black hole size differs from that in four-dimensional general relativity case due to the non-linear term in the effective Einstein equation. In the negative cosmological constant case, we obtain an implication on the Newton constant in the Karch-Randall model.

I. INTRODUCTION

The braneworld model is a phenomenological model which describes our four-dimensional universe in higher-dimensional theory. In this model, we are living on a four-dimensional membrane, and only gravity propagates to the extra dimension. Among several models for braneworlds, Randall-Sundrum (RS) type models are interesting because they provide us many phenomenological predictions [1–3]. In these models, extra dimension is warped due to the self-gravity of the branes. Because of this warping, it is found in some RS type models that gravity can be confined near the brane and becomes four-dimensional even when the extra dimension is non-compact [4, 5].

Although many studies on the RS model have been done, there are still some open issues. One of them is that static solutions of black holes localized on the brane are missing. Though numerical solutions of such brane-localized black holes are constructed when the black hole size is smaller than the bulk curvature scale [6, 7], no solutions are found when the size is large. For this issue, the following conjecture has been proposed based on the adS/CFT correspondence [8–10] (see also Ref. [11] for related issues). According to the correspondence, a five-dimensional classical brane-localized black hole is dual to a four-dimensional black hole that emits the Hawking radiation. Since the latter one cannot be static due to the Hawking radiation emission, it is suggested by the duality that there is no static brane-localized black hole which is larger than the bulk curvature radius.

Here, one might realize that the adS/CFT correspondence also tells that static solutions may present when the black hole horizon is extreme [12] since the horizon temperature is zero and the Hawking radiation will not be emitted. Indeed, the authors of Ref. [12] constructed the near-horizon geometry of such extreme charged static black hole localized on the asymptotically flat brane and studied its properties. In this paper, we shall consider the near-horizon geometry of extreme charged black hole localized on the brane with non-vanishing cosmological constant to study the properties of the brane-localized black holes in more generalized settings. We also intend to reveal the non-trivial property of the gravity in the braneworld model with negative cosmological constant, the Karch-Randall model.

The rest of this paper is organized as follows. In Sec. II, we describe the model we study. We sketch the metric form for the near-horizon geometry in Sec. III and we present numerical solutions in Sec. IV. In Sec. V, we give analytic arguments for relatively large black holes. Finally, we give summary and discussion in Sec. VI.

II. MODELS

The model we consider in this paper is the RS braneworld model, which consists of five-dimensional asymptotically anti-de Sitter (adS) bulk spacetime and a four-dimensional brane with positive tension in it. The action of this model is given by

$$
\begin{aligned}
S \;=\; & \frac{1}{2\kappa_5^2}\int_M d^5x\sqrt{-g}\left({}^{(5)}R+\frac{12}{l^2}\right) \\
& +\frac{1}{\kappa_5^2}\int_{\partial M} d^4x\sqrt{-h}K \\
& +\int_{\mathrm{brane}} d^4x\sqrt{-h}\left(-\sigma-\frac{1}{2\kappa_4^2}F_{\mu\nu}F^{\mu\nu}\right), \qquad (1)
\end{aligned}
$$

where M is the bulk spacetime and ∂M is its outer boundary. $h_{\mu\nu}$ is the induced metric on the brane.

*Electronic address: ryotaku@tap.scphys.kyoto-u.ac.jp
†Electronic address: shiromizu@tap.scphys.kyoto-u.ac.jp
‡Electronic address: tanahasi@yukawa.kyoto-u.ac.jp

$\kappa_5^2 = 8\pi G_5$ and $\kappa_4^2 = 8\pi G_4$ are the five and four-dimensional gravitational coupling, respectively. l is the bulk curvature radius. σ and $F_{\mu\nu}$ are the brane tension and the field strength of the Maxwell field on the brane. K is the trace of the extrinsic curvature $K_{\mu\nu}$ of ∂M. We impose the Z_2-symmetry about the brane.

From the above action, we obtain the five-dimensional Einstein equation in the bulk as

$$R_{MN} - \frac{1}{2}Rg_{MN} = \frac{4}{l^2}g_{MN}. \tag{2}$$

Under the Z_2-symmetry, the Israel's junction condition on the brane is given by [13]

$$K_{\mu\nu} - Kh_{\mu\nu} = \frac{1}{2}\kappa_5^2 T_{\mu\nu}, \tag{3}$$

where $T_{\mu\nu}$ is the energy-momentum tensor on the brane, which is given as

$$T_{\mu\nu} = -\sigma h_{\mu\nu} + \frac{2}{\kappa_4^2}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}F^2 h_{\mu\nu}\right). \tag{4}$$

The Maxwell equation and the Bianchi equation are

$$d * F = 0, \quad dF = 0, \tag{5}$$

where * is the Hodge dual in four dimensions.

III. NEAR-HORIZON GEOMETRY, BULK EQUATIONS AND BOUNDARY CONDITIONS

A. Near-horizon geometry

We consider a static brane-localized black hole whose horizon is made to be extreme by the Maxwell field on the brane. A static black hole has constant surface gravity on its horizon. Then, when the horizon is extremal on the intersection with the brane, the whole part of the horizon in the bulk will also be extremal. For such an extremal horizon, we can take the near-horizon limit and analyze its properties. It is proved that the near-horizon geometry of a static extreme black hole can be written in a warped product of a two-dimensional Lorentzian space and a compact manifold as [14]

$$ds^2 = A(x)^2 d\Sigma^2 + g_{ab}dx^a dx^b, \tag{6}$$

where $d\Sigma^2$ is a two-dimensional Lorentzian metric M_2 of constant curvature $2k$. When the metric describes the black hole spacetime, k should be negative and then M_2 is two-dimensional AdS spacetime (adS$_2$). We also assume that $g_{ab}dx^a dx^b$ has $SO(3)$ symmetry. Choosing the coordinates $x^a = (\rho, \theta, \phi)$, the near-horizon geometry becomes

$$ds^2 = A(\rho)^2 d\Sigma^2 + d\rho^2 + R(\rho)^2 d\Omega^2, \tag{7}$$

where $d\Omega^2$ is the metric of the two-dimensional unit sphere.

B. Bulk equations

For the metric ansatz of Eq. (7), the bulk Einstein equations, Eq. (2), becomes

$$\frac{k}{A^2} - \frac{A'^2}{A^2} - \frac{2A'R'}{AR} - \frac{A''}{A} = -\frac{4}{l^2}, \tag{8}$$

$$\frac{A''}{A} + \frac{R''}{R} = \frac{2}{l^2} \tag{9}$$

and

$$\frac{1}{R^2} - \frac{R'^2}{R^2} - \frac{2A'R'}{AR} - \frac{R''}{R} = -\frac{4}{l^2}, \tag{10}$$

where prime stands for the derivative with respect to ρ. From these we obtain

$$\frac{k}{A^2} + \frac{1}{R^2} = \frac{A'^2}{A^2} + \frac{R'^2}{R^2} + \frac{4A'R'}{AR} - \frac{6}{l^2}, \tag{11}$$

which is the Hamiltonian constraint.

We assume the horizon to be compact, which implies that $R(\rho)$ vanishes somewhere. Then, we set the "origin" of ρ as $R(\rho = 0) = 0$. The smoothness of the horizon at the "origin" requires $R'(0) = 1$ and $A'(0) = 0$. Then, the only free parameter under the boundary condition at $\rho = 0$ is $A(0) = A_0$. After all, the bulk equations have three free parameters $\{A_0, k, l\}$.

Here, note that the equations have two families of scaling invariance: $(A, k) \to (\lambda_1 A, \lambda_1^2 k)$ and $(R, l, \rho, k) \to (\lambda_2 R, \lambda_2 l, \lambda_2 \rho, \lambda_2^{-2} k)$. Then, we can set $A_0 = 1$ and $l = 1$ without loss of generality, and then, the only free parameter will be k. After getting a solution $(\tilde{A}(\tilde{\rho}), \tilde{R}(\tilde{\rho}))$, we can recover a dimensionful solution as $(A_0\tilde{A}(l^{-1}\rho), l\tilde{R}(l^{-1}\rho))$.

C. Junction condition

From Eq. (3), the junction condition determines the extrinsic curvature $K_{\mu\nu}$ on the brane as

$$K_{\mu\nu}|_{\rm brane} = \frac{\kappa_5^2\sigma}{6}h_{\mu\nu} + \frac{\kappa_5^2}{\kappa_4^2}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}F^2 h_{\mu\nu}\right). \tag{12}$$

The induced cosmological constant on the brane Λ_4 is given as [21]

$$\Lambda_4 \equiv -\frac{3}{l^2} + \frac{\kappa_5^4\sigma^2}{12}. \tag{13}$$

From this expression, however, we see that Λ_4 is bounded from below as $\Lambda_4 \geq -3/l$. In Ref. [12], the brane tension is tuned to make the brane asymptotically flat. In our current paper, we will not impose such tuning. Then, the brane geometry will be asymptotically de Sitter, anti-de Sitter or Minkowski spacetimes depending the value of

Λ_4. For convenience, we introduce the following dimensionless parameter

$$\alpha \equiv \frac{\sigma}{\sigma_{RS}}, \tag{14}$$

where $\sigma_{RS} \equiv 6/\kappa_5^2 l$ is the value of the tension when the brane geometry is asymptotically Minkowski spacetime. $\alpha = 1$ corresponds to $\Lambda_4 = 0$. By the definition of α and Λ_4, they are related as

$$\alpha = \sqrt{1 + \frac{l^2\Lambda_4}{3}} = \frac{l\kappa_5^2\sigma}{6}. \tag{15}$$

Note that $\alpha > 1$ ($\alpha < 1$) for $\Lambda_4 > 0$ ($\Lambda_4 < 0$).

Now, we suppose that the brane is located at $\rho = \rho_0$. Then the Israel junction condition (12) becomes

$$\frac{A(\rho_0)'}{A(\rho_0)} = \alpha - \frac{\kappa_5^2}{\kappa_4^2}\frac{Q^2}{2L_2^4}, \quad \frac{R(\rho_0)'}{R(\rho_0)} = \alpha + \frac{\kappa_5^2}{\kappa_4^2}\frac{Q^2}{2L_2^4}. \tag{16}$$

Here, we used a notation for the induced metric on the brane such that

$$ds^2_{\rm brane} = |k|L_1^2 d\Sigma^2 + L_2^2 d\Omega^2, \tag{17}$$

where L_1 and L_2 are proper radii of M_2 and S^2 defined by

$$L_1^2 \equiv |k|^{-1}A(\rho_0)^2, \quad L_2^2 \equiv R(\rho_0)^2. \tag{18}$$

Moreover, we used the solution for the Maxwell field

$$*F = Qd\Omega, \tag{19}$$

where Q is the total charge on the brane given by

$$Q = \frac{1}{4\pi}\int_{S^2} *F. \tag{20}$$

From Eq. (11) and the junction condition, we have

$$\begin{aligned} \frac{\mathrm{sign}(k)}{L_1^2} + \frac{1}{L_2^2} &= \frac{6}{l}(\alpha^2 - 1) - \frac{\kappa_5^4}{\kappa_4^4}\frac{Q^4}{2L_2^8} \\ &= 2\Lambda_4 - \frac{\kappa_5^4}{\kappa_4^4}\frac{Q^4}{2L_2^8}, \end{aligned} \tag{21}$$

where $\mathrm{sign}(k)$ is equal to 1, 0 or -1 when k is positive, zero or negative. From Eq. (21), we find some restrictions on the near-horizon geometry. When $\alpha \le 1$, k is always negative. When $\alpha > 1$, on the other hand, k can be positive for some large enough values of α.

D. Gravitational couplings

Here, we would like to make a comment on the relation between the four and five-dimensional gravitational couplings. From several analyses [15], it is sure that the relation for the cases with $\Lambda_4 \ge 0$ ($\alpha \ge 1$) is given by

$$\kappa_4^2 = \frac{\kappa_5^4\sigma}{6} = \frac{\kappa_5^2\alpha}{l}. \tag{22}$$

On the other hand, we do not have a definite answer for the case of $\Lambda_4 < 0$. This case is called Karch-Randall model. When adS$_4$ curvature radius scale is sufficiently larger than the bulk curvature scale l, however, it is expected that $\kappa_4^2 \approx \kappa_5^2/l$ holds [16]. For the moment, we will use the relation $\kappa_4^2 = \kappa_5^2/l$ for all ranges of Λ_4. We will ask this issue again in Sec. V D.

IV. THE SOLUTIONS

Let us solve the bulk equations from $\rho = 0$ to $\rho = \rho_0$ for fixed values of k. In this section, we employ the unit of $l = 1$ and also set $A_0 = 1$. From the Israel junction condition (16), Q and α are determined as

$$\alpha = \frac{1}{2}\left(\frac{A'}{A} + \frac{R'}{R}\right)|_{\rho=\rho_0} \tag{23}$$

and

$$Q^2 = \frac{\kappa_4^2}{\kappa_5^2}R^4\left(\frac{R'}{R} - \frac{A'}{A}\right)|_{\rho=\rho_0}. \tag{24}$$

As shown in Ref. [12], there are analytic solutions of Eqs. (8)-(10) for some special values of k. One of them is

$$A(\rho) = 1, \quad R(\rho) = \frac{1}{\sqrt{2}}\sinh(\sqrt{2}\rho) \tag{25}$$

for $k = -4$. $k = -1$ yields another exact solution as

$$A(\rho) = \cosh\rho, \quad R(\rho) = \sinh\rho. \tag{26}$$

For these exact solutions, the geometry on the brane is somewhat restricted. Substituting the above solutions into Eq. (23), we find

$$\alpha = \frac{1}{\sqrt{2}}\coth(\sqrt{2}\rho_0) \overset{\rho_0\to\infty}{=} \frac{1}{\sqrt{2}}, \tag{27}$$

for $k = -4$, and

$$\alpha = \frac{1}{2}(\tanh\rho_0 + \coth\rho_0) > 1 \tag{28}$$

for $k = -1$. From Eq. (28), we see that there are no solutions for which $\alpha < 1$ when $k = -1$. That is, we cannot obtain a brane with $\Lambda_4 \le 0$ in this case. On the other hand, when $k = -4$, we see from Eq. (27) that we can realize a brane with $\Lambda_4 < 0$ if we choose sufficiently large ρ_0.

For general k, the bulk solution behaves as follows.

1. $k < -4$ case: $A(\rho)$ monotonically decreases and vanishes at a point ρ_1. At this point, $R(\rho)$ diverges and a curvature singularity appears. Therefore, the brane position ρ_0 must be smaller than ρ_1.

2. $-4 < k < -1$ case: Both $A(\rho)$ and $R(\rho)$ increase exponentially. α has a minimum between $1/\sqrt{2}$ and 1, and tends to 1 for $\rho_0 \to \infty$.

3. $-1 < k$ case: Both $A(\rho)$ and $R(\rho)$ increase exponentially. α decreases monotonically and tends to 1 for $\rho_0 \to \infty$.

We show the behaviours of $A(\rho)$ and $R(\rho)$ in Fig. 1 and that of α in Fig. 2. Solutions for $k = 0$ is not black hole solutions, while they are limiting solutions for black hole solution sequences with $k < 0$.

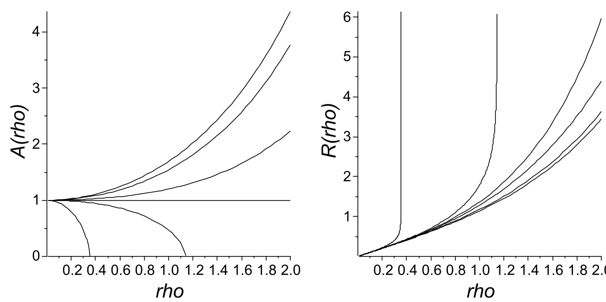

FIG. 1: Profiles of $A(\rho)$ and $R(\rho)$. In the left panel for $A(\rho)$, the curves from top to bottom represent the solutions for $k = 0, -1, -3, -4, -6$ and -32, respectively. For $R(\rho)$, the curves from bottom to top are for $k = 0, -1, -3, -4, -6$ and -32. When $k < -4$, a solution has a singularity.

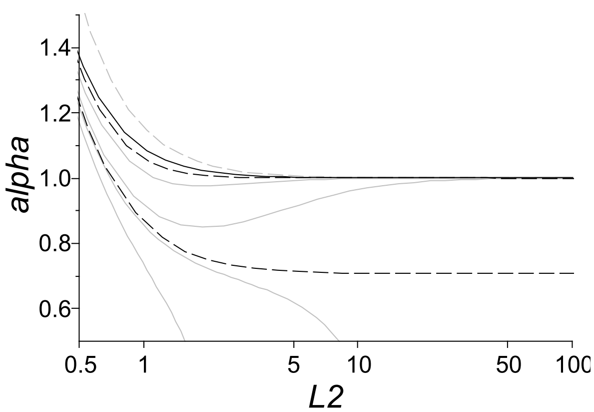

FIG. 2: $L_2 = R(\rho_0)$ dependence of α. L_2 is the horizon radius on the brane for each k. The dark solid line corresponds to the $k = 0$ solution. The lines that run above it, including the light dashed line for $k = 100$, are for $k > 0$. Those solutions for $k > 0$ do not represent black holes. The dark-dashed lines represent the solutions for $k = -1$ and -4 from the top, and the light-solid lines represent those for $k = -2, -3.5, -4.1$ and -5 from the top.

From Fig. 2, we can see that there is an upper bound on the four-dimensional horizon size on the brane, $L_2 = R(\rho_0)$, for $\alpha > 1$. Such an upper bound on the horizon size also appears in the ordinary general relativity for black holes in the de Sitter universe [17–19]. We will examine this feature later in Sec. V B.

Next, we study the ratio between the five-dimensional and four-dimensional black hole entropies. This ratio should become the unity if the bulk/boundary correspondence works, and this expectation is confirmed to be correct for the flat brane case in the large black hole limit [12]. We would like to extend this study on the duality to the non-flat brane case.

The five and four-dimensional black hole entropies are defined as

$$S_5 = \frac{(\text{Area of 5D horizon})}{4G_5} = \frac{2\pi}{G_5}\int_0^{\rho_0} R(\rho)^2 d\rho \quad (29)$$

and

$$S_4 = \frac{(\text{Area of 4D horizon})}{4G_4} = \frac{\pi}{G_4} R(\rho_0)^2, \quad (30)$$

respectively. The ratio between them is given by

$$\frac{S_5}{S_4} = \frac{G_4}{G_5}\frac{2}{R(\rho_0)^2}\int_0^{\rho_0} R(\rho)^2 d\rho. \quad (31)$$

We show $R(\rho_0)$ dependence of entropy ratio in Fig. 3. The upper panel is of the solutions for $\alpha \leq 1$ with asymptotically adS or Minkowski branes, and the lower is for $\alpha > 1$ with asymptotically de Sitter branes. As we explained in Sec. III D, We used $G_4/G_5 = 1$ in the plot for $\alpha < 1$ while $G_4/G_5 = \alpha$ in the plot for $\alpha > 1$. In the $\alpha > 1$ case, we can see that the ratio tends to the unity if the four-dimensional black hole radius L_2 is larger than the bulk curvature scale. In the $\alpha \leq 1$ case, on the other hand, the ratio tends to some constant smaller than the unity as L_2 becomes large. We will study on these properties again in Secs. V B, V C and V D.

V. LARGE BLACK HOLE LIMIT

When the black hole radius is much larger than the bulk curvature scale l, the brane is near the bulk conformal boundary and then the behaviour of the black hole on the brane is expected to coincide with one in the ordinary four-dimensional general relativity. In this section, we will discuss the large black hole limits with partial help of numerical analysis. Through out this section, we set $l = 1$.

A. Some basics: metric and extrinsic curvature

When $k > -4$, both $A(\rho)$ and $R(\rho)$ behave like e^ρ for large ρ, as we showed in Sec. IV. So we can write $A \to A_\infty e^\rho$, $R \to R_\infty e^\rho$. A_∞ and R_∞ are determined by solving the equations for each k. Then, the metric becomes

$$ds^2 \simeq d\rho^2 + e^{2\rho} R_\infty^2 (a^2 d\Sigma^2 + d\Omega^2), \quad (32)$$

where $a \equiv A_\infty / R_\infty$, which is a function of k. Note that we can know its function form only after solving the bulk equation numerically from $\rho = 0$ to $\rho = \rho_0$. Fig. 4 shows

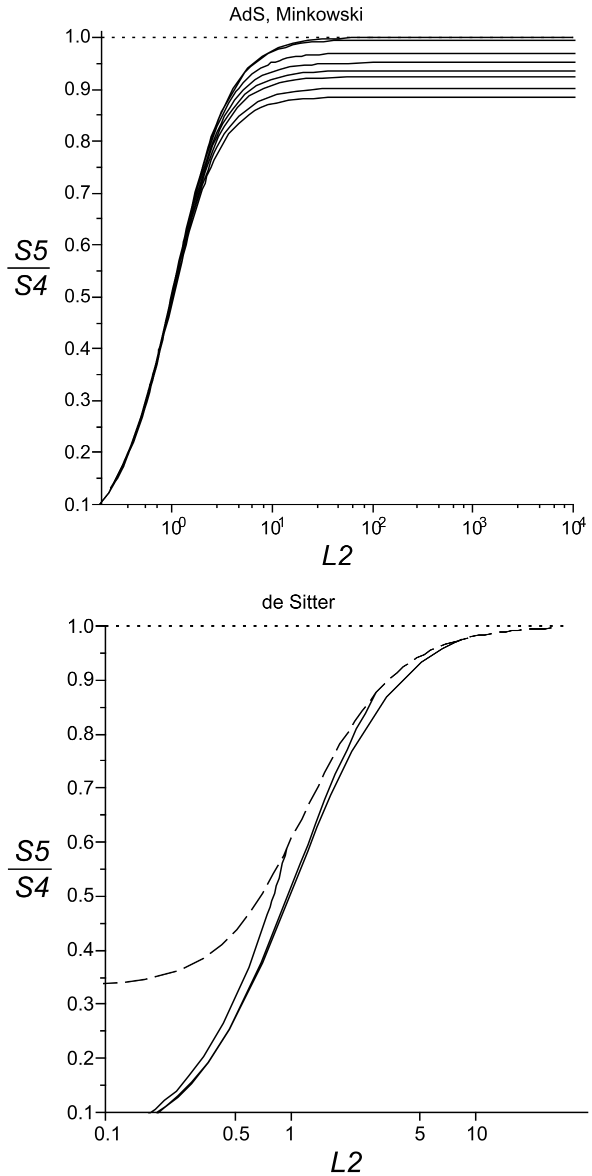

FIG. 3: $L_2 = R(\rho_0)$ dependence of the entropy ratio S_5/S_4. The upper panel is for the solutions for $\alpha \leq 1$ with asymptotically adS brane. Each lines correspond to $\alpha = 1, 0.9995, 0.995, 0.99, 0.985, 0.98, 0.97$ from 0.96 from top to bottom. The lower panel is for the solutions for $\alpha > 1$ with asymptotically de Sitter or flat brane. The dashed line is $k = 0$ case. The solid lines correspond to $\alpha = 1.001, 1.01$ and 1.1 from right to left.

the k dependence of a^2. a^2 converges to a positive constant for $k \to 0$ and approaches zero as $k \to -4$.

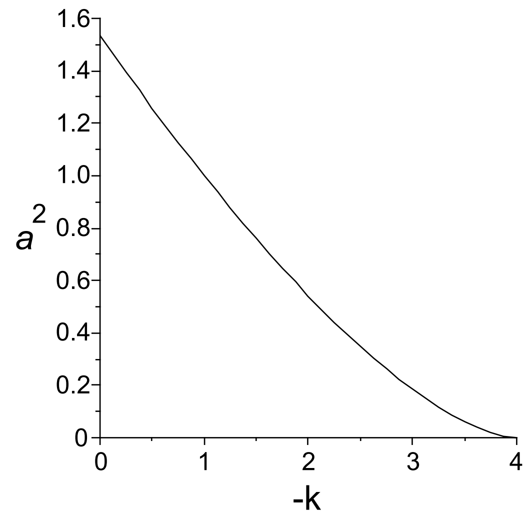

FIG. 4: k dependence of a^2. a^2 converges to a finite value $a(k=0) \sim 1.53419$ as $k \to 0$, becomes unity for $k = -1$ as suggested by Eq. (26), and approaches zero for $k \to -4$.

Let us introduce a new convenient coordinate defined by $r \equiv r_0 e^{-2\rho}$ to solve the five-dimensional Einstein equation approximately. In this coordinate, the conformal boundary is at $r = 0$. The metric is written as

$$\begin{aligned} ds^2 &= \frac{dr^2}{4r^2} + A(r)^2 d\Sigma + R(r)^2 d\Omega^2 \\ &\simeq \frac{dr^2}{4r^2} + \frac{R_\infty^2 r_0}{r}(a^2 d\Sigma^2 + d\Omega^2). \end{aligned} \tag{33}$$

Hereafter, we take $r_0 = R_\infty^{-2}$ for convenience and we will focus on $r = \epsilon \ll 1$ limit. Following Ref. [20], we obtain the analytic solutions for $A(r)$ and $R(r)$ near the conformal boundary as

$$\begin{aligned} A(r)^2 &= \frac{a^2}{r}\Bigg[1 + \Big(\frac{1}{6} - \frac{k}{3a^2}\Big)r - \frac{1}{48}\Big(1 - \frac{k^2}{a^4}\Big)r^2 \log r \\ &\quad + \Big(\frac{5}{288} - \frac{k}{36a^2} + \frac{5k^2}{288a^4} + \lambda\Big)r^2 + \cdots\Bigg] \end{aligned} \tag{34}$$

and

$$\begin{aligned} R(r)^2 &= \frac{1}{r}\Bigg[1 - \left(\frac{1}{3} - \frac{k}{6a^2}\right) r + \frac{1}{48}\left(1 - \frac{k^2}{a^4}\right) r^2 \log r \\ &\quad + \left(\frac{5}{288} - \frac{k}{36a^2} + \frac{5k^2}{288a^4} - \lambda\right) r^2 + \dots\Bigg], \end{aligned} \tag{35}$$

where λ is an integral constant determined by k. Then, the extrinsic curvature

$$K_{\mu\nu} dx^\mu dx^\nu = K_1 d\Sigma^2 + K_2 d\Omega^2 \tag{36}$$

is computed as

$$\begin{aligned} K_1 &= a^2\Bigg[\frac{1}{r} + \frac{1}{48}\left(1 - \frac{k^2}{a^4}\right) r \log r + \Bigg(\frac{1}{48}\left(1 - \frac{k^2}{a^4}\right) \\ &\quad - \frac{5}{288} + \frac{k}{36a^2} - \frac{5k^2}{288a^4} - \lambda\Bigg) r + \cdots\Bigg] \end{aligned} \tag{37}$$

and

$$K_2 = \frac{1}{r} - \frac{1}{48}\left(1-\frac{k^2}{a^4}\right) r\log r + \left(-\frac{1}{48}\left(1-\frac{k^2}{a^4}\right) - \frac{5}{288} + \frac{k}{36a^2} - \frac{5k^2}{288a^4} + \lambda\right) r + \cdots . \quad (38)$$

Using K_1 and K_2, Eqs. (23) and (24) are rewritten as

$$\alpha = \frac{1}{2}\left(\frac{K_1}{A(\epsilon)^2} + \frac{K_2}{R(\epsilon)^2}\right) \quad (39)$$

and

$$Q^2 = \frac{\kappa_4^2}{\kappa_5^2} R(\epsilon)^4 \left(\frac{K_2}{R(\epsilon)^2} - \frac{K_1}{A(\epsilon)^2}\right). \quad (40)$$

B. $\alpha > 1$ case: de Sitter brane

In $\alpha > 1$ case, positive cosmological constant is induced on the brane and the brane geometry becomes asymptotically de Sitter spacetime. From Fig. 2, we can see that there is a restriction on the black hole size in the sense that the black hole size $R(\rho_0)$ has an upper bound which depends on α. The size of the black hole horizon in de Sitter spacetime is known to be restricted by the cosmological constant in the ordinary general relativity [17–19]. From our result, we can confirm that the same restriction holds even in the braneworld setup.

Comparing the braneworld upper limit $\alpha_{\rm max}^{\rm BW}$ with the upper limit $\alpha_{\rm max}^{\rm 4D}$ in the ordinary four-dimensional general relativity (see Fig. 5), we can see that

$$\alpha_{\rm max}^{\rm BW} > \alpha_{\rm max}^{\rm 4D} = \sqrt{1+\frac{1}{6L_2^2}} \quad (41)$$

is satisfied. It tells us that the restriction on the black hole size is weaker in the braneworld model. The value of $\alpha_{\rm max}^{\rm BW}$ is given by $k=0$ solution.

Let us study the difference between $\alpha_{\rm max}^{\rm BW}$ and $\alpha_{\rm max}^{\rm 4D}$ in more detail. First of all, we check that $\alpha_{\rm max}^{\rm 4D}$ is smaller than $\alpha_{\rm max}^{\rm BW}$. To focus on $\alpha_{\rm max}^{\rm BW}$, we set k to zero. Then, we see from Eqs. (34), (35), (37) and (38) that

$$\frac{K_1}{A^2} = 1 - \frac{1}{6}\epsilon + \frac{1}{24}\epsilon^2\log\epsilon + \left(\frac{1}{72} - 2\lambda\right)\epsilon^2 + \mathcal{O}(\epsilon^3\log\epsilon) \quad (42)$$

and

$$\frac{K_2}{R^2} = 1 + \frac{1}{3}\epsilon - \frac{1}{24}\epsilon^2\log\epsilon + \left(\frac{1}{18} + 2\lambda\right)\epsilon^2 + \mathcal{O}(\epsilon^3\log\epsilon). \quad (43)$$

Then, Eqs. (39) yields

$$\alpha_{\rm max}^{\rm BW} = 1 + \frac{1}{12}\epsilon + \frac{5}{144}\epsilon^2 + \mathcal{O}\left(\epsilon^3\log\epsilon\right). \quad (44)$$

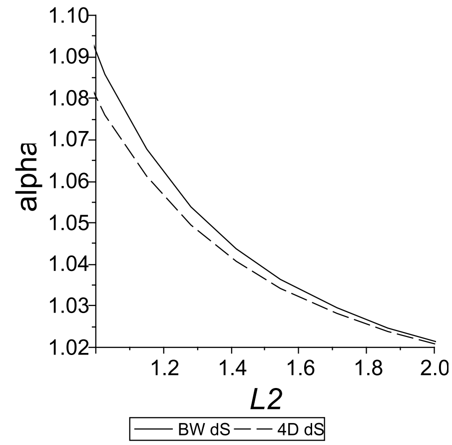

FIG. 5: Dependence of $\alpha_{\rm max}^{\rm BW}$ (solid line) and $\alpha_{\rm max}^{\rm 4D}$ (dashed line) about the horizon radius L_2.

To find the expression for $\alpha_{\rm max}^{\rm 4D}$, we should find that for L_2 first. Since $-k/a^2 \to 0$ for $k \to 0$, we find from Eq. (35) that

$$L_2^2 = R^2(\rho_0) = \frac{1}{\epsilon} - \frac{1}{3} + \frac{1}{48}\epsilon\log\epsilon + \mathcal{O}(\epsilon). \quad (45)$$

Replacing L_2 by ϵ in Eq. (41), we find the expression of $\alpha_{\rm max}^{\rm 4D}$ as

$$\alpha_{\rm max}^{\rm 4D} = \sqrt{1+\frac{1}{6L_2^2}} = 1 + \frac{1}{12}\epsilon + \frac{7}{288}\epsilon^2 + \mathcal{O}\left(\epsilon^3\log\epsilon\right). \quad (46)$$

Then, we can confirm that $\alpha_{\rm max}^{\rm 4D}$ is smaller than $\alpha_{\rm max}^{\rm BW}$:

$$\alpha_{\rm max}^{\rm BW} - \alpha_{\rm max}^{\rm 4D} = \frac{1}{96}\epsilon^2 + \mathcal{O}(\epsilon^3\log\epsilon). \quad (47)$$

Next, we give an interpretation of the difference between them using the effective Einstein equations [21]. The trace of effective Einstein equations becomes

$$-{}^{(4)}R = -4\Lambda_4 + \frac{Q^4}{(\alpha_{\rm max}^{\rm BW})^2 L_2^8} \quad (48)$$

and we can see that the non-linear term, $Q^4/(\alpha_{\rm max}^{\rm BW})^2L_2^8$, weakens the effect of the cosmological constant. This non-linear term is evaluated in terms of ϵ as follows. From Eqs. (42) and (43), we find that Q^2 of Eq. (40) becomes

$$Q^2 = \frac{1}{2\epsilon} - \frac{1}{12}\log\epsilon + \mathcal{O}(1). \quad (49)$$

Then, we find from Eqs. (44), (45) and (49) that

$$\frac{Q^2}{\alpha_{\rm max}^{\rm BW} L_2^4} = \frac{1}{2}\epsilon - \frac{1}{12}\epsilon^2\log\epsilon + \mathcal{O}\left(\epsilon^2\right). \quad (50)$$

From Eq. (48), we can read off the difference between $\Lambda_4^{\rm 4D}$ and $\Lambda_4^{\rm BW}$ as

$$\delta\Lambda_4 \simeq -\frac{Q^4}{4(\alpha_{\rm max}^{\rm BW})^2 L_2^8} \simeq -\frac{1}{16L_2^4}. \quad (51)$$

In the above, we used Eqs. (45) and (50). This matches with the value that is evaluated from from Eqs. (44) and (46) at the leading order, which is given as

$$\Lambda_4^{4\mathrm{D}} - \Lambda_4^{\mathrm{BW}} = 3\Big((\alpha_{\max}^{4\mathrm{D}})^2 - (\alpha_{\max}^{\mathrm{BW}})^2\Big) \simeq -\frac{1}{16L_2^4}. \quad (52)$$

C. $\alpha < 1$ case: anti-de Sitter brane

In this subsection, we consider $\alpha < 1$ case in which the brane geometry is asymptotically anti-de Sitter spacetime. This is the so called Karch-Randall (KR) model [5]. In this model, it is expected that the relation $G_4/G_5 = 1$ holds approximately when the four-dimensional adS curvature radius L is sufficiently larger than the bulk curvature scale. In this section, we fix $G_4/G_5 = 1$ for any L though we guess that this relation does not hold in general. We will address this issue later in Sec. V D.

We compare the braneworld solution with four-dimensional extreme anti-de Sitter Reissner-Nordström (adS-RN) solution in the general relativity which share the same horizon radius L_2 and four-dimensional cosmological constant Λ_4. For sufficiently large L_2, the adS$_2$ radius for this adS-RN solution, $L_{1(4\mathrm{D})}$, becomes (see Appendix A)

$$L_{1(4\mathrm{D})}{}^2 = \frac{L_2^2}{1-2\Lambda_4 L_2^2} \simeq \frac{1}{-2\Lambda_4} = \frac{1}{6(1-\alpha^2)}. \quad (53)$$

Fig. 6 shows L_2 dependence of $L_1 = |k|^{-1/2}A(\rho_0)$ for the braneworld black hole solutions. From this, we can see that the size of adS$_2$, L_1, tends to the values of four-dimensional adS-RN black hole, $L_{1(4\mathrm{D})}$, when Λ_4 is sufficiently close to zero.

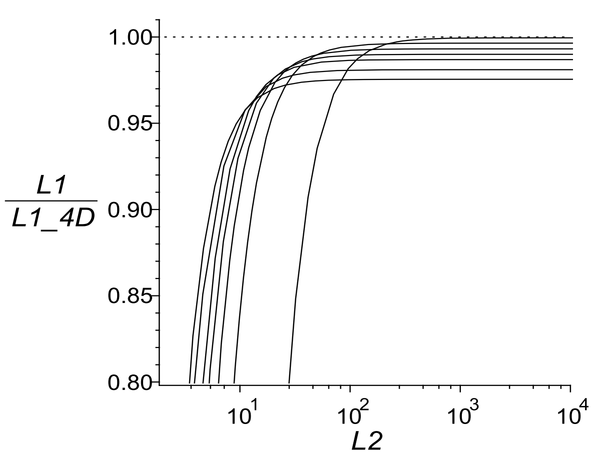

FIG. 6: L_2 dependence of $L_1 = |k|^{-1/2}A(\rho_0)$ for fixed values of α. L_1 is normalised by $L_{1(4\mathrm{D})}$. The lines from top to bottom at large L_2 regime are for $\alpha = 0.9995, 0.995, 0.99, 0.985, 0.98, 0.97$ and 0.96, respectively.

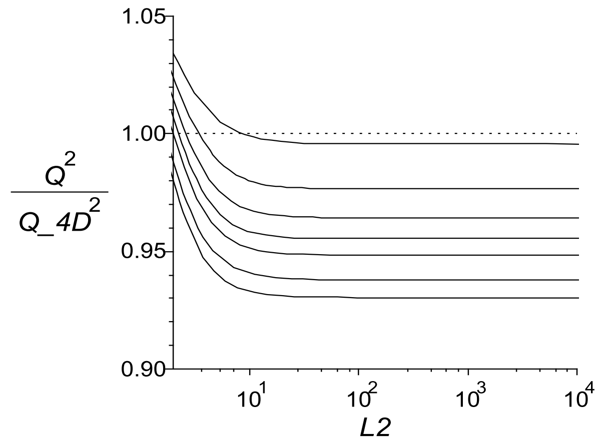

FIG. 7: L_2 dependence of Q^2/Q_{4D}^2. The lines from top to bottom are for $\alpha = 0.9995, 0.995, 0.99, 0.985, 0.98, 0.97$ and 0.96, respectively.

Charge Q for the extreme adS-RN black hole in the general relativity is given in terms of L_2 as

$$Q_{4\mathrm{D}}{}^2 = -L_2^4\Lambda_4 + L_2^2. \quad (54)$$

Let us compare it with that of the braneworld black hole. We show L_2 dependence of $Q/Q_{4\mathrm{D}}$ in Fig. 7. In this figure, we see that Q and $Q_{4\mathrm{D}}$ coincide when Λ_4 is sufficiently close to zero. Thus, the large braneworld extreme black hole has the same near-horizon geometry as four-dimensional adS-RN black hole in the limit of vanishing Λ_4. This figure also suggests that the discrepancy between Q and $Q_{4\mathrm{D}}$ can be non-zero when Λ_4 is non-zero, and it becomes a constant independent of L_2.

Using the large black hole limit, we can evaluate L_1 and Q in terms of $1-\alpha$, that is, in terms of Λ_4. We consider the case that both of $l \ll L$ and $l \ll L_2$ holds, where l and $L \equiv (-3/\Lambda_4)^{1/2}$ are five and four-dimensional curvature scales and L_2 is four-dimensional horizon size on the brane. We expect that the bulk/brane duality would work under these conditions. In the following, we focus on $L \ll L_2$ regime (see Appendix B 1 for the results in $L \gg L_2$ regime).

In the $L \ll L_2$ regime, $L_1 \sim \mathcal{O}\left((1-\alpha)^{-1}\right)$ holds as we can see in Fig. 6 and Eq. (53). Then, from Eqs. (34), (35) and the definitions of L_1 and L_2, we find $\epsilon \ll -k\epsilon/a^2 \sim 1-\alpha \ll 1$. This regime is realized in the limit of $k \to -4$, for which a tends to zero as shown in Fig. 4. After some calculations in this regime, we find for $\epsilon \to 0$ that (see Appendix B 2 for derivations of the following equations)

$$\frac{L_1^2}{L_{1(4\mathrm{D})}{}^2} = 1 - \frac{3}{2}(1-\alpha) + \mathcal{O}\big((1-\alpha)^2\log(1-\alpha)\big). \quad (55)$$

From this equation, as $\alpha \to 1$, we can see that L_1^2 approaches that of the four-dimensional extreme adS-RN solution, which is given by Eq. (53).

We can analyze behaviour of the charge Q in the same way. The result is

$$\frac{Q^2}{{Q_{4\mathrm{D}}}^2} = 1 + 2(1-\alpha)\log(1-\alpha) + \mathcal{O}(1-\alpha), \qquad (56)$$

and we find that Q approaches that of the four-dimensional adS-RN solution, Eq. (54), in the limit of $\alpha \to 1$.

D. Gravitational coupling for adS branes

If the adS/CFT correspondence holds in the KR model, it is natural to expect that $S_5 = S_4$ holds, at least in the large black hole limit. In this paper, however, we observed that $S_5 \neq S_4$ in that limit when we suppose $G_4/G_5 = 1$. In this subsection, we would like to propose a formula for G_4/G_5 which makes S_5 equal to S_4 for any Λ_4.

In Fig. 8, we show α dependence of the ratio S_5/S_4 in the large black hole limit. This entropy ratio is proportional to G_4/G_5 as shown in Eq. (31). Then, the value of G_4/G_5 that makes the entropy ratio to be unity will be inverse of the value of S_5/S_4 shown in Fig. 8.

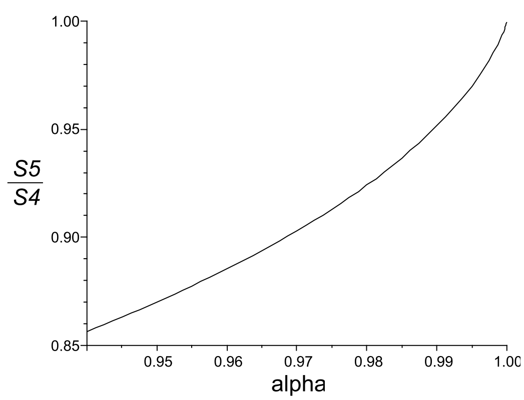

FIG. 8: α dependence of the entropy ratio S_5/S_4 calculated in the large black hole limit $\rho_0 \to \infty$.

Let us study this value of G_4/G_5 in the limit of $\alpha \to 1$. In this limit, we can expand A_5 as (see Appendix B 2)

$$A_5 = \frac{4\pi}{\epsilon}\Big(1 + 2(1-\alpha)\log(1-\alpha) + \mathcal{O}\left(1-\alpha\right)\Big). \qquad (57)$$

Then, we obtain

$$\begin{aligned}\frac{S_5}{S_4} &= \frac{G_4}{G_5}\frac{A_5}{A_4} \\ &= \frac{G_4}{G_5}\Big(1 + 2\left(1-\alpha\right)\log\left(1-\alpha\right) + \mathcal{O}\left(1-\alpha\right)\Big). \qquad (58)\end{aligned}$$

In order that this ratio is equal to unity, we should set G_5/G_4 as

$$\begin{aligned}\frac{G_5}{G_4} &= 1 + 2(1-\alpha)\log(1-\alpha) + \cdots \\ &= 1 + \frac{l^2}{L^2}\log\left(\frac{2l^2}{L^2}\right) + \cdots . \qquad (59)\end{aligned}$$

Since the charge Q is also proportional to G_4/G_5, as seen in Eq. (24), we may determine the value of G_4/G_5 requiring that the charge ratio Q/Q_{4D} becomes unity as $\alpha \to 1$. Interestingly, the expression of G_4/G_5 determined in this way coincides with Eq. (59) at least up to sub-leading order in the $\alpha \to 1$ limit (see Eq. (56)). This fact may imply that G_4 and G_5 should be related by Eq. (59) for general α, that is, for general Λ_4.

VI. SUMMERY AND DISCUSSION

In this paper, we analyzed the near-horizon geometry of charged extreme black holes localized on the brane with non-vanishing cosmological constant in the RS-type braneworld models. In the de Sitter brane case, we find that there is an upper bound on the black hole size and that the bound is determined by the cosmological constant on the brane. This restriction on the horizon size also appeared in the ordinary four-dimensional general relativity, while the restriction was found to be weaker in the braneworld case due to the non-linear term in the effective Einstein equation on the brane.

In the anti-de Sitter brane case, we observed discrepancies between the near-horizon geometry of the brane-localized black hole from that of the four-dimensional extreme adS-RN black hole. We found that the adS$_2$ radius and the charge are smaller than those of four-dimensional adS-RN black holes, and confirmed that those discrepancies vanish in the flat brane limit ($\alpha \to 1$). We also calculated the five and four-dimensional black hole entropies assuming $G_4/G_5 = 1$. As a result, it turned out that S_5/S_4 becomes smaller as the cosmological constant on the brane becomes larger.

In the Karch-Randall model, it is suggested by Ref. [16] that the four-dimensional gravity weakens as $G_4/G_5 \approx 1 - \mathcal{O}\left(l^2/L^3\right)\mathcal{R}$ for $L \lesssim \mathcal{R} \lesssim L^3/l^2$, where $\mathcal{R}$ is the separation of two gravitating objects, due to small four-dimensional graviton mass. However, the formula for G_4/G_5 we proposed in this paper, Eq. (59), has a different form from it. It may be peculiar that our formula is independent of the black hole size, while the formula of Ref. [16] depends on propagation distance $\mathcal{R}$ of the gravity. It will be interesting to investigate whether these two formulae are compatible or not.

There are many remaining issues. In our work, we addressed the near-horizon geometry only. To justify our result, we have to construct the full bulk solutions. Perturbative approaches for the solution construction like Ref. [22] or numerical solution construction methods like Ref. [6] may give fruitful results. In Ref. [7],

it is pointed out that nonsystematic error increases as taking the asymptotic boundary farther from the horizon even if the black hole radius is smaller than the AdS curvature radius, which could imply the singularity formation in the bulk. It is also valuable to examine whether such nonsystematic error exist in the extremal case. Another interesting subject is the near-horizon geometry of a rotating extreme black hole localized on the brane. Such a black hole has the spontaneous emission through the superradiant modes [24], although its temperature is zero. Thus, the adS/CFT correspondence about the braneworld models may suggest that such a black hole will be a dynamical. It is meaningful to address if it is true or not. These studies will be helpful for understanding on the black hole solutions in higher-dimensional spacetime models and also on the adS/CFT correspondence in generalized situations.

Acknowledgments

We would like to thank Roberto Emparan, Shunichiro Kinoshita, Takashi Nakamura and Takahiro Tanaka for useful discussions. This work was supported by the Grant-in-Aid for the Global COE Program"The Next Generation of Physics, Spun from Universality and Emergence" from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. TS is partially supported by Grant-Aid for Scientific Research from Ministry of Education, Science, Sports and Culture of Japan (Nos. 21244033, 21111006, 20540258 and 19GS0219), the Japan-U.K. Research Cooperative Programs.

Appendix A: 4D Reissner-Nordström black hole

In this appendix, we summarise the fundamental features of the extreme static charged black hole solutions in the four-dimensional ordinary general relativity. We use this solution as a fiducial to compare with the brane-localized charged black holes in this paper.

The metric of charged black hole solutions in the four-dimensional ordinary general relativity with a cosmological constant Λ_4 is given as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{A1}$$

where

$$f(r) = 1 - \frac{\Lambda_4}{3}r^2 + \frac{Q^2}{r^2} - \frac{2M}{r}. \tag{A2}$$

The horizon radius r_H is determined by $f(r_H) = 0$, which implies

$$M = \frac{1}{2}\Big(r_H + \frac{Q^2}{r_H} - \frac{\Lambda_4}{3}r_H^3\Big). \tag{A3}$$

When the black hole is extreme, $f'(r_H) = 0$ holds. In this case, we find

$$\Lambda_4 r_H^4 - r_H^2 + Q^2 = 0. \tag{A4}$$

One of roots for this is given by

$$r_H^2 = \frac{1}{2\Lambda_4}\left(1 - \sqrt{1 - 4\Lambda_4 Q^2}\right), \tag{A5}$$

and $r = r_H$ determined by this equation will be the black hole horizon. Now, $f(r)$ is written as

$$f(r) = (r - r_H)^2 \times \frac{g(r)}{r^2}, \tag{A6}$$

where

$$g(r) \equiv 1 - \frac{\Lambda_4}{3}\left(r^2 + 2r_H r + 3r_H^2\right). \tag{A7}$$

If $\Lambda_4 > 0$, the equation $f'(r) = 0$ has another positive root $r = \tilde{r}_H$, which is given by

$$\tilde{r}_H^2 = \frac{1}{2\Lambda_4}\left(1 + \sqrt{1 - 4\Lambda_4 Q^2}\right). \tag{A8}$$

The surface $r = \tilde{r}_H$ is, however, not the black hole horizon because $g(\tilde{r}_H) < 0$. It is rather the cosmological horizon of the de Sitter universe.

The near-horizon geometry of this extreme black hole is given by

$$ds^2 \simeq \frac{r_H^2}{g(r_H)}\left(-x^2 dt'^2 + \frac{dx^2}{x^2}\right) + r_H^2 d\Omega^2, \tag{A9}$$

where we introduced new coordinates as $x = r - r_H$ and $t' = \frac{g(r_H)}{r_H}t$. As is well-known, this geometry is $\mathrm{adS}_2 \times \mathrm{S}^2$. The radius of each submanifold is given by

$$L_1^2 = \frac{r_H^2}{g(r_H)} = \frac{r_H^2}{\sqrt{1 - 4\Lambda_4 Q^2}} = \frac{L_2^2}{1 - 2\Lambda_4 L_2^2} \tag{A10}$$

and

$$L_2^2 = r_H^2. \tag{A11}$$

Appendix B: Large black hole limit in anti-de Sitter case

In this section, we give a detailed analysis on the large black hole limit in adS brane case of Sec. V C. We focus on a regime in which both of $L \gg l$ and $L_2 \gg l$ are satisfied, i.e., the regime in which the adS/CFT correspondence would work, and investigate on $L_2 \ll L$ and $L \ll L_2$ cases in subsections B 1 and B 2, respectively. We set $l = 1$ in this section unless otherwise noted.

To facilitate the following analysis, we introduce

$$\gamma \equiv -\frac{k}{a^2}, \qquad \delta \equiv \gamma - 1, \qquad \beta \equiv 1 - \alpha. \tag{B1}$$

Note that γ and δ are functions of k. δ becomes zero for $k=-1$, increases monotonically as k decreases, and diverges as $k \to -4$, as we can see from Fig. 4. In the case of adS brane with $L \gg l$ and $L_2 \gg l$, we may assume that ϵ and β are positive value much smaller than the unity. In this case, we find from Eq. (39) that

$$\begin{aligned}\beta &= \frac{\gamma-1}{12}\epsilon - \frac{5\gamma^2+8\gamma+5}{144}\epsilon^2 + \mathcal{O}\big(\gamma^3\epsilon^3\log\epsilon, \epsilon^3\log\epsilon\big)\\ &= \frac{1}{12}\delta\epsilon - \frac{5\delta^2+18\delta+18}{144}\epsilon^2 + \mathcal{O}\big(\delta^3\epsilon^3\log\epsilon, \epsilon^3\log\epsilon\big),\end{aligned} \tag{B2}$$

where we used Eqs. (34), (35), (37) and (38) and assumed $\gamma\epsilon \ll 1$ so that the expansion converges. We treat $\delta \ll 1$ and $\delta \gg 1$ cases separately in the following.

1. $\delta \ll 1$ case and $L_2 \ll L$ regime

For $\delta \ll 1$, dominant part of Eq. (B2) is given by

$$\beta = \frac{1}{12}\delta\epsilon - \frac{1}{8}\epsilon^2 + \mathcal{O}\left(\delta\epsilon^2, \epsilon^3\log\epsilon\right). \tag{B3}$$

This equation in terms of ϵ has two roots for $\beta < \delta^2/72$, and they are given by

$$\epsilon = \epsilon_\pm \equiv \frac{1}{3}\left(\delta \pm \sqrt{\delta^2 - 72\beta}\right). \tag{B4}$$

Let us inspect ϵ_+ first. It behaves as $\epsilon_+ \sim 2\delta/3$ when $\beta \ll \delta^2$. Since $L_2 \simeq l/\sqrt{\epsilon}$ and $L \simeq l/\sqrt{2\beta}$, $(L_2/l)^2 \ll L/l$ and thus $L_2 \ll L$ follows in this regime, that is, the four-dimensional black hole radius on the brane becomes much smaller than the four-dimensional curvature scale when $\beta \ll \delta^2$. In this regime, it is convenient to parametrize the deviation of ϵ_+ from $2\delta/3$ as

$$\delta \equiv \frac{3}{2}(1+\chi)\,\epsilon, \tag{B5}$$

where we assume $0 < \chi \ll 1$. In this notation, ϵ is related to β as

$$\beta = \frac{1}{12}\chi\epsilon^2 + \mathcal{O}\left(\epsilon^3\log\epsilon\right). \tag{B6}$$

The χ term in the right-hand side will be dominant over $\mathcal{O}(\epsilon^3\log\epsilon)$ term if $\chi \gg \epsilon\log\epsilon$. We find in this regime that the expansion forms of L_1^2, L_2^2 and Q^2 become

$$L_1^2 = \frac{1}{\epsilon} - 1 - \frac{3}{2}\chi + \left(\frac{33}{16} + \lambda + \frac{17}{4}\chi\right)\epsilon + \mathcal{O}\left(\epsilon^2\log\epsilon\right), \tag{B7}$$

$$L_2^2 = \frac{1}{\epsilon} - \frac{1}{2} - \left(\frac{3}{16} + \lambda + \frac{1}{4}\chi\right)\epsilon + \mathcal{O}\left(\epsilon^2\log\epsilon\right), \tag{B8}$$

$$Q^2 = \frac{1}{\epsilon} - \frac{1}{4} + 4\lambda + \frac{3}{4}\chi + \mathcal{O}\left(\epsilon\log\epsilon\right). \tag{B9}$$

λ in the above is a function of k and ϵ, i.e., a function of δ and ϵ, and it is determined so that the bulk geometry becomes regular. Since we know that the bulk metric reduce to that of adS_5 in the limit of $\chi \to 0$ and $\epsilon \to 0$, we can fix the leading term of λ as (see also ref. [12])

$$\lambda|_{\epsilon=0=\chi} = 0. \tag{B10}$$

Note that ϵ or χ may appear in the sub-leading terms of λ. Then, we find the correct expansion forms of L_1^2, L_2^2 and Q^2 to be

$$L_1^2 = \frac{1}{\epsilon} - 1 - \frac{3}{2}\chi + \frac{33}{16}\epsilon + \mathcal{O}\left(\lambda\epsilon, \chi\epsilon, \epsilon^2\log\epsilon\right), \tag{B11}$$

$$L_2^2 = \frac{1}{\epsilon} - \frac{1}{2} - \frac{3}{16}\epsilon + \mathcal{O}\left(\lambda\epsilon, \chi\epsilon, \epsilon^2\log\epsilon\right), \tag{B12}$$

$$Q^2 = \frac{1}{\epsilon} - \frac{1}{4} + \mathcal{O}\left(\lambda, \chi, \epsilon\log\epsilon\right). \tag{B13}$$

These expression coincide with those for flat brane case given in [12] in the limit of $\chi \to 0$, and difference appears only in L_1 up to the order shown here. We have to clarify sub-leading behavior of λ to fix the higher order terms of these expansion equations, while it seems difficult to do it analytically.

Next, we make some comments on another solution ϵ_-. Let us fix β and consider δ dependence of ϵ_-. Fixing β, we can show that ϵ_- monotonically decreases as we increase δ, and ϵ_- takes the maximum value $\delta/3$ for $\delta = 6\sqrt{2\beta}$. This behaviour can be expressed equivalently as $L/l \lesssim (L_2/l)^2$, that is, the brane black hole size is of the same order as or larger than the four-dimensional curvature scale. The brane black hole size L_2 grows as $L_2 \sim 12\beta/\delta$ for $\delta^2 \gg \beta$. This branch of solution is smoothly connected to that for $\delta \gg 1$, and we will analyze it in the next subsection in detail.

2. $\delta \gg 1$ case and $L \ll L_2$ regime

We will focus on the case δ and then γ is much larger than the unity in the aim of studying the black holes much larger than the four-dimensional curvature scale. We use γ instead of δ throughout this subsection.

Before solving Eq. (B2) to find the black hole radius, we fix the leading behaviour of λ from the bulk regularity. From Eqs. (15), (34), (35), (37), (38) and (39), we find

$$\begin{aligned}\frac{L_1^2}{{L_{1(4\mathrm{D})}}^2} = 1 &- \frac{1}{8}\gamma\epsilon - \frac{1}{6}\gamma^2\epsilon^2\log\epsilon - \frac{1}{4}\epsilon - \frac{1}{8}\frac{\epsilon}{\gamma}\\ &- 8\lambda\epsilon^2 + \mathcal{O}\left(\gamma^2\epsilon^2, \gamma\epsilon^2\log\epsilon\right),\end{aligned} \tag{B14}$$

where $L_{1(4\mathrm{D})}$ is the radius of four-dimensional adS-RN solution, which is given by Eq. (A10). For a fixed $\gamma\epsilon$, this ratio should converge to some constant in the limit of $\epsilon \to 0$, as we can see in Fig. 6, while the third term in the right-hand side, $\gamma^2\epsilon^2\log\epsilon$, diverges in such a limit. We have only $\lambda\epsilon^2$ term to cancel such divergence. This

λ is a function of γ and ϵ, and may have the following leading behaviour:

$$\lambda = \frac{\gamma^2}{48}\log\gamma. \tag{B15}$$

This λ replaces the logarithmic term as $\log\epsilon \to \log\gamma\epsilon$ and the divergence is canceled. We use this leading form of λ henceforth.

When $\delta \gg 1$, the expression of β, Eq. (B2), can be expressed as

$$\beta = \frac{\tilde{\epsilon}}{12} - \frac{\epsilon}{12} - \frac{5}{144}\tilde{\epsilon}^2 - \frac{1}{8}\epsilon\tilde{\epsilon} + \mathcal{O}\left(\tilde{\epsilon}^3\log\epsilon, \epsilon^2, \lambda\tilde{\epsilon}\epsilon^2\right), \tag{B16}$$

where we introduced $\tilde{\epsilon} \equiv \gamma\epsilon$. Note that $\epsilon \ll \tilde{\epsilon} \ll 1$ by assumption and then $\beta \sim \tilde{\epsilon} \gg \epsilon$, i.e., $L_2 \gg L$ follows in this regime. Solving Eq. (B16) as an equation of $\tilde{\epsilon}$ and expanding it with respect to β and ϵ, we find a solution which satisfies the condition $\tilde{\epsilon} \ll 1$ as

$$\tilde{\epsilon} = 12\beta + 60\beta^2 + (1 + 18\beta)\,\epsilon + \mathcal{O}\left(\beta^3, \epsilon^2\right). \tag{B17}$$

Plugging this expression into Eq. (B14), we find

$$\frac{L_1^2}{{L_{1(4\mathrm{D})}}^2} = 1 - \frac{3}{2}\beta - \frac{3}{8}\epsilon + \mathcal{O}\left(\beta^2\log\beta, \frac{\epsilon^2}{\beta}\right), \tag{B18}$$

and we obtain Eq. (55) by taking ϵ to zero while keeping β fixed. To proceed the expansion and determine the higher-order terms, we have to know the sub-leading behaviour of λ, while it seems not straightforward.

In a similar manner, we can evaluate the ratio of Q of a brane-localized black hole to $Q_{4\mathrm{D}}$ of the four-dimensional adS-RN solution, which is given by Eq. (54). Using Eqs. (B15) and (B16), we obtain the expansion form of $Q^2/{Q_{4\mathrm{D}}}^2$, Eq. (56), as

$$\begin{aligned}\frac{Q^2}{{Q_{4\mathrm{D}}}^2} &= 1 + \frac{1}{6}\tilde{\epsilon}\log\tilde{\epsilon} - \frac{1}{6}\epsilon\log\tilde{\epsilon} + \mathcal{O}\left(\tilde{\epsilon}\right) \\ &= 1 + 2\beta\log\beta + \mathcal{O}\left(\beta, \epsilon\right). \end{aligned}\tag{B19}$$

Finally, let us calculate the five-dimensional horizon area and its ratio to the four-dimensional horizon area on the brane. The area of five-dimensional horizon is given as

$$\begin{aligned} A_5 &= 2\int_0^{\rho(r=\epsilon)} 4\pi R\left(\rho\right)^2 d\rho \\ &= 8\pi\int_0^{\rho(r=1/\gamma)} R\left(\rho\right)^2 d\rho + 4\pi\int_\epsilon^{1/\gamma}\frac{R(r)^2}{r}dr, \end{aligned}\tag{B20}$$

where we divided the integral into two pieces for convenience of the following calculation. Note that $\epsilon \ll 1/\gamma \ll 1$ by assumption. Using $R(\rho) \simeq R_\infty^2 e^{2\rho}$ and $R^2 \simeq 1/r$, which hold for $\rho \gg 1$ and $r \ll 1$, the first integral in the right-hand side of Eq. (B20) becomes

$$8\pi\int_0^{\rho(r=1/\gamma)} R\left(\rho\right)^2 d\rho \simeq 4\pi R^2|_{r=1/\gamma} = \mathcal{O}\left(\gamma\right) = \mathcal{O}\left(\frac{\tilde{\epsilon}}{\epsilon}\right). \tag{B21}$$

Using Eqs. (35) and (B15), the second integral of Eq. (B20) becomes

$$\begin{aligned} &4\pi\int_\epsilon^{1/\gamma}\frac{R(r)^2}{r}dr \\ &= 4\pi\int_\epsilon^{1/\gamma} dr\left(\frac{1}{r^2} - \frac{2+\gamma}{6r} + \mathcal{O}\left(\gamma^2\log(\gamma r), r\right)\right) \\ &= 4\pi\left[-\frac{1}{r} - \frac{2+\gamma}{6}\log r + \mathcal{O}\left(\gamma^2 r\log\left(\gamma r\right)\right)\right]_\epsilon^{1/\gamma} \\ &= \frac{1}{\epsilon}\left(1 + \frac{2\epsilon+\tilde{\epsilon}}{6}\log\tilde{\epsilon} + \mathcal{O}\left(\tilde{\epsilon}\right)\right). \end{aligned}\tag{B22}$$

Since the first integral, Eq. (B21), can be absorbed in $\mathcal{O}\left(\tilde{\epsilon}\right)$ of Eq. (B22), we find that A_5 is given by Eq. (B22). Writing it in terms of β and ϵ, we obtain

$$A_5 = \frac{4\pi}{\epsilon}\left(1 + 2\beta\log\beta + \frac{1}{2}\epsilon\log\beta + \mathcal{O}\left(\beta\right)\right), \tag{B23}$$

and it gives Eq. (57) for $\epsilon \to 0$. It is straightforward to calculate the ratio between the five and four-dimensional horizon areas, $A_5/A_4 = A_5/(4\pi L_2^2)$. Using Eqs. (B23) and (35), we find

$$\frac{A_5}{A_4} = 1 + 2\beta\log\beta + \frac{1}{2}\epsilon\log\beta + \mathcal{O}\left(\beta\right). \tag{B24}$$

This yields Eq. (58) for $\epsilon \to 0$.

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [arXiv:hep-ph/9905221].

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].

[3] For review, R. Maartens and K. Koyama, arXiv:1004.3962.

[4] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000) [arXiv:hep-th/9911055].

[5] A. Karch and L. Randall, JHEP **0105**, 008 (2001) [arXiv:hep-th/0011156].
[6] H. Kudoh, T. Tanaka and T. Nakamura, Phys. Rev. D **68**, 024035 (2003) [arXiv:gr-qc/0301089].
[7] H. Yoshino, JHEP **0901**, 068 (2009) [arXiv:0812.0465 [gr-qc]].
[8] R. Emparan, A. Fabbri and N. Kaloper, JHEP **0208**, 043 (2002) [arXiv:hep-th/0206155].
[9] T. Tanaka, Prog. Theor. Phys. Suppl. **148**, 307 (2003) [arXiv:gr-qc/0203082].
[10] R. Emparan, J. Garcia-Bellido and N. Kaloper, JHEP **0301**, 079 (2003) [arXiv:hep-th/0212132].
[11] N. Tanahashi and T. Tanaka, Prog. Theor. Phys. **123** (2010) 369 [arXiv:0910.5303 [gr-qc]]; K. Kashiyama, N. Tanahashi, A. Flachi and T. Tanaka, JHEP **2010** (2010) 099 [arXiv:0910.5376 [gr-qc]].
[12] A. Kaus and H. S. Reall, JHEP **0905**, 032 (2009) [arXiv:0901.4236 [hep-th]].
[13] W. Israel, Nuovo Cim. B **44S10**, 1 (1966) [Erratum-ibid. B **48**, 463 (1967 NUCIA,B44,1.1966)].
[14] H. K. Kunduri, J. Lucietti and H. S. Reall, Class. Quant. Grav. **24**, 4169 (2007) [arXiv:0705.4214 [hep-th]].
[15] For example, U. Gen and M. Sasaki, Prog. Theor. Phys. **108**, 471 (2002) [arXiv:gr-qc/0201031].
[16] N. Kaloper and L. Sorbo, JHEP **0508**, 070 (2005) [arXiv:hep-th/0507191].
[17] T. Shiromizu, K. Nakao, H. Kodama and K. I. Maeda, Phys. Rev. D **47**, 3099 (1993).
[18] S. A. Hayward, T. Shiromizu and K. i. Nakao, Phys. Rev. D **49**, 5080 (1994) [arXiv:gr-qc/9309004].
[19] K. Maeda, T. Koike, M. Narita and A. Ishibashi, Phys. Rev. D **57**, 3503 (1998) [arXiv:gr-qc/9712029].
[20] S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. **217**, 595 (2001) [arXiv:hep-th/0002230].
[21] T. Shiromizu, K. i. Maeda and M. Sasaki, Phys. Rev. D **62**, 024012 (2000) [arXiv:gr-qc/9910076].
[22] H. Kodama, Prog. Theor. Phys. **120**, 371 (2008) [arXiv:0804.3839 [hep-th]].
[23] R. Emparan, G. T. Horowitz and R. C. Myers, JHEP **0001**, 007 (2000) [arXiv:hep-th/9911043].
[24] For example, O. J. C. Dias, R. Emparan and A. Maccarrone, Phys. Rev. D **77**, 064018 (2008) [arXiv:0712.0791 [hep-th]].